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2008 J. Phys. A: Math. Theor. 41 355209

(http://iopscience.iop.org/1751-8121/41/35/355209)

J. Phys. A: Math. Theor. **41** (2008) 355209 (14pp) doi:10.1088/1751-8113/41/35/355209

Solving the hierarchy of the nonisospectral KdV equation with self-consistent sources via the inverse scattering transform

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Received 15 May 2008, in final form 30 June 2008
Published 28 July 2008
Online at stacks.iop.org/JPhysA/41/355209

Abstract
The hierarchy of the nonisospectral KdV equation with self-consistent sources is derived. N-soliton solutions of the hierarchy are obtained through the inverse scattering transform. We develop new treatment of singularities appearing in the Lax pair in the process of determining time revolutions of scattering data. Our approach is general and can apply to other (1 + 1)-dimensional nonisospectral soliton hierarchy with self-consistent sources.

PACS numbers: 02.30.Ik, 05.45.Yv

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent years, the soliton equations with self-consistent sources [1–3] were studied for both physical and mathematical interests. In general, the sources can result in solitary waves moving with nonconstant velocity and cause a great variety of dynamics of soliton solutions. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves [4]. Besides, the sources can lead to ghost solitons [5, 6]. Recently, many classical methods, such as the inverse scattering transform (IST), Darboux transformation and bilinear approach, have been used to find exact solutions for soliton equations with self-consistent sources [7–14].

The nonisospectral soliton equations usually describe solitary waves in nonuniform media [15–17]. These equations are related to time-dependent spectral parameters with polynomial evolution $\lambda_t = \sum \alpha_j \lambda^j$ in which each coefficient α_j corresponds to a certain type of relaxation effect. For example, one Hirota–Satsuma equation, the KdV equation with loss and nonuniformity terms, is related to $\lambda_t = -2\alpha\lambda$, and α describes the relaxation and nonuniformity of the media [16]. Besides, the nonisospectral flows related to $\lambda_t = \lambda^j$ usually play the role of master symmetries and then generate time-dependent symmetries [18, 19].

In this paper, we solve the following hierarchy of the nonisospectral KdV equation with self-consistent sources (KdVESCS) through the IST,

$$u_t = T^n(xu_x + 2u) - 2\sum_{j=1}^{N}\left(\phi_j^2\right)_x, \qquad (n = 0, 1, 2, \ldots), \tag{1.1a}$$

$$\phi_{j,xx} = (\lambda_j - u)\phi_j, \qquad (j = 1, 2, \ldots, N), \tag{1.1b}$$

where $\phi_j \doteq \phi_j(x,t)$, $\lambda_j = \kappa_j^2 > 0$, $\kappa_j > 0$, $(j = 1, 2, \ldots, N)$ are distinct and

$$T = \partial^2 + 4u + 2u_x\partial^{-1}, \qquad \partial = \frac{\partial}{\partial x}, \qquad \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right)\mathrm{d}x. \tag{1.2}$$

This hierarchy corresponds to a time-dependent spectral parameter λ with time evolutions $\lambda_t = \frac{1}{2}(4\lambda)^{n+1}$, and therefore can be used to describe the solitary waves in a certain type of nonuniform media. The way to derive the hierarchy and its Lax pair is similar to that in [8, 12]. We develop the treatment of singularities appearing in the Lax pair in the process of determining time revolutions of scattering data. This treatment is similar to [8]. Our approach can apply to other (1 + 1)-dimensional nonisospectral soliton hierarchy with self-consistent sources.

This paper is organized as follows. In section 2, the hierarchy of the nonisospectral KdVESCS is derived. In section 3, some results of the direct scattering problem are listed briefly for completeness. Section 4 determines the time evolutions of the scattering data, and finally section 5 gives the N-soliton solutions of the hierarchy and discusses dynamics for some obtained solutions.

2. The hierarchy of the nonisospectral KdVESCS

In this section, we derive the hierarchy of the nonisospectral KdVESCS. Our approach is directly on the basis of the eigenfunctions of recursion operator and differs slightly from the one using constrained flows [7, 8].

Let us start from the following linear problems:

$$\phi_{xx} = (\lambda - u)\phi, \tag{2.1a}$$

$$\phi_t = A\phi + B\phi_x. \tag{2.1b}$$

From the related compatibility condition $\phi_{xx,t} = \phi_{t,xx}$, we have

$$A = -\tfrac{1}{2}B_x, \tag{2.2a}$$

$$u_t = \tfrac{1}{2}TB_x - 2\lambda B_x + \lambda_t, \tag{2.2b}$$

where T is defined by (1.2). Now we expand B as

$$B = \sum_{k=0}^{n} b_k\lambda^{n-k} + \sum_{j=1}^{N}\frac{\mu_j}{\lambda - \lambda_j}, \tag{2.3a}$$

with

$$B|_{u=0} = 4^n x, \tag{2.3b}$$

where $\{\lambda_j\}$ are N distinct eigenvalues satisfying (1.1b). Substituting (2.3a) into (2.2b) and taking $\lambda_t = \frac{1}{2}(4\lambda)^{n+1}$ yield

$$u_t = T^n(xu_x + 2u) - 2\sum_{j=1}^{N} \mu_{j,x}, \qquad (n = 0, 1, 2, \ldots), \tag{2.4a}$$

$$T\mu_{j,x} = 4\lambda_j \mu_{j,x}, \qquad (j = 1, 2, \ldots, N). \tag{2.4b}$$

Then, by taking

$$\mu_j = \phi_j^2, \tag{2.5}$$

where ϕ_j satisfies (1.1b), we can reach the hierarchy of the nonisospectral KdVESCS (1.1). Obviously, it is easy to obtain the Lax pair of the hierarchy (1.1), i.e., (2.1), where

$$A = -\frac{1}{2}(4\lambda)^n - \sum_{j=1}^{n}(4\lambda)^{n-j}T^{j-1}(xu_x + 2u) - \frac{1}{2}\sum_{j=1}^{N}\frac{(\phi_j^2)_x}{\lambda - \lambda_j}, \tag{2.6a}$$

$$B = (4\lambda)^n x + \sum_{j=1}^{n} 2(4\lambda)^{n-j}\partial^{-1}T^{j-1}(xu_x + 2u) + \sum_{j=1}^{N}\frac{\phi_j^2}{\lambda - \lambda_j}. \tag{2.6b}$$

The first three equations in the hierarchy (1.1a) are

$$u_t = xu_x + 2u - 2\sum_{j=1}^{N}(\phi_j^2)_x, \tag{2.7a}$$

$$u_t = xK_1 + 4u_{xx} + 8u^2 + 2u_x\partial^{-1}u - 2\sum_{j=1}^{N}(\phi_j^2)_x, \tag{2.7b}$$

$$u_t = xK_2 + 6K_{1,x} + 12uu_{xx} + 32u^3 + 2K_1\partial^{-1}u + 6u_x\partial^{-1}u^2 - 2\sum_{j=1}^{N}(\phi_j^2)_x, \tag{2.7c}$$

where

$$K_1 = u_{xxx} + 6uu_x, \qquad K_2 = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x,$$

and each $\{K_j\}$ are the isospectral KdV flows described by

$$K_j = T^j u_x, \qquad (j = 1, 2, \ldots). \tag{2.8}$$

3. The direct scattering problem

For completeness, we briefly list some results of the direct scattering problem related to the spectral problem (2.1a) [20, 21]. We replace λ with $-k^2$ in the spectral problem (2.1). We also suppose that $u(x,t)$ and $\{\phi_j(x,t)\}\,(j = 1, 2, \ldots, N)$ vanish rapidly as $|x| \to \infty$, and the real potential $u(x,t)$ satisfies $\int_{-\infty}^{+\infty}|x^j u(x)|\,\mathrm{d}x < \infty\,(j = 0, 1, 2)$.

There are two Jost solutions for the spectral problem (2.1a), $\phi^+(x,k)$ and $\phi^-(x,k)$, which are bounded for all values of x and are analytic on $\mathrm{Im}\,k > 0$ and continuous on $\mathrm{Im}\,k \geqslant 0$ for k. These two Jost solutions satisfy the asymptotic conditions,

$$\phi^+(x,k) \sim \mathrm{e}^{\mathrm{i}kx}, \qquad \phi_x^+(x,k) \sim \mathrm{i}k\,\mathrm{e}^{\mathrm{i}kx}, \qquad (x \to \infty) \tag{3.1a}$$

$$\phi^-(x,k) \sim \mathrm{e}^{-\mathrm{i}kx}, \qquad \phi^-_x(x,k) \sim -\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}kx}, \qquad (x \to -\infty). \tag{3.1b}$$

For all real value k, there are linear relationships between two Jost solutions

$$\phi^-(x,k) = a(k)\phi^+(x,-k) + b(k)\phi^+(x,k), \tag{3.2}$$

where

$$a(k) = \frac{1}{2\mathrm{i}k} W(\phi^-(x,k),\phi^+(x,k)), \qquad b(k) = \frac{1}{2\mathrm{i}k} W(\phi^+(x,-k),\phi^-(x,k)), \tag{3.3}$$

and $a(k)$ is analytic on $\mathrm{Im}\,k > 0$ and continuous on $\mathrm{Im}\,k \geqslant 0$, while $b(k)$ is only defined on $\mathrm{Im}\,k = 0$. Here $W(\mu,\nu)$ is a Wronskian defined by $W(\mu,\nu) = \mu\nu_x - \nu\mu_x$.

The function $a(k)$ has a finite number of simple zeros at $k_1, k_2, \ldots, k_N$ on imaginary axis of the upper half k-plane, i.e. $k_m = \mathrm{i}\kappa_m, \kappa_m > 0, (m = 1, 2, \ldots, N)$. So $\phi^+(x, \mathrm{i}\kappa_m)$ and $\phi^-(x,\mathrm{i}\kappa_m)$ are linearly dependent.

Scattering data for the spectral problem (2.1) are defined by

$$S(t) : \left\{ R(k) = \frac{b(k)}{a(k)}, \mathrm{Im}\,k = 0, \mathrm{i}\kappa_m, c_m, m = 1, 2, \ldots, N \right\}, \tag{3.4}$$

where each c_m is the so-called normalization constant of eigenfunction $\phi^+(x,\mathrm{i}\kappa_m)$, defined by $c_m^2 = -\frac{\mathrm{I}b_m}{a_k(\mathrm{i}\kappa_m)}$, $b_m = \frac{\phi^-(x,\mathrm{i}\kappa_m)}{\phi^+(x,\mathrm{i}\kappa_m)}$.

4. The time evolutions of the scattering data

In this section, we derive the time evolutions of the scattering data.

By $\phi(x,t,\mathrm{i}\kappa_m)$ denote the normalized Jost solution of $\phi^+(x,t,k)$ with $k = \mathrm{i}\kappa_m$ and the normalization constant $c_m(t)$, i.e.,

$$\phi(x,t,\mathrm{i}\kappa_m) = c_m(t)\phi^+(x,t,\mathrm{i}\kappa_m), \qquad (m = 1, 2, \ldots, N). \tag{4.1}$$

Meanwhile, corresponding $\lambda_j = -(\mathrm{i}\kappa_j)^2$ in (1.1*b*), we can set $\phi_j \doteq \phi_j(x,t)$ in (1.1*b*) to be

$$\phi_j = \sqrt{2\beta_j(t)}\phi(x,t,\mathrm{i}\kappa_j), \qquad (j = 1, 2, \ldots, N), \tag{4.2}$$

where $\beta_j(t)$ is an arbitrary positive function of t. Thus we have

$$\beta_j(t) = \frac{1}{2}\int_{-\infty}^{\infty} \phi_j^2\,\mathrm{d}x. \tag{4.3}$$

Then, combining (1.1*b*) with $\lambda_j = -(\mathrm{i}\kappa_j)^2$ and (2.1*a*) with $\lambda = -k^2$ leads to the following:

$$\phi_{j,xx}\phi(x,t,k) - \phi_j\phi_{xx}(x,t,k) = \left(k^2 + \kappa_j^2\right)\phi_j\phi(x,t,k), \tag{4.4}$$

which further means

$$(\phi_{j,x}\phi(x,t,k) - \phi_j\phi_x(x,t,k))_x = \left(k^2 + \kappa_j^2\right)\phi_j\phi(x,t,k), \tag{4.5a}$$

$$(\phi_{j,x}\phi(x,t,k) - \phi_j\phi_x(x,t,k))_{x,k} = 2k\phi_j\phi(x,t,k) + \left(k^2 + \kappa_j^2\right)\phi_j(\phi(x,t,k))_k. \tag{4.5b}$$

Integrating the above equalities and taking $k \to \mathrm{i}\kappa_m$ at mean time, we have

$$\phi_{j,x}\phi(x,t,\mathrm{i}\kappa_m) - \phi_j\phi_x(x,t,\mathrm{i}\kappa_m) = \left(\kappa_j^2 - \kappa_m^2\right)\partial^{-1}(\phi_j\phi(x,t,\mathrm{i}\kappa_m)), \tag{4.6a}$$

$$\phi_{j,x}(\phi(x,t,\mathrm{i}\kappa_m))_k - \phi_j(\phi(x,t,\mathrm{i}\kappa_m))_{x,k} = 2\mathrm{i}\kappa_m\partial^{-1}(\phi_j\phi(x,t,\mathrm{i}\kappa_m)), \tag{4.6b}$$

where $\partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right)\mathrm{d}x$, and these two equalities are valid for arbitrary j and m. Then taking $x \to \infty$ in (4.6) yields

$$\int_{-\infty}^{\infty} \phi_j \phi(x,t,\mathrm{i}\kappa_m)\,\mathrm{d}x = 0, \qquad (j \neq m), \tag{4.7a}$$

$$\int_{-\infty}^{\infty} \phi_j \phi(x,t,\mathrm{i}\kappa_m)\partial^{-1}(\phi_j\phi(x,t,\mathrm{i}\kappa_m))\,\mathrm{d}x = 0, \qquad (j = 1,2,\ldots,N), \tag{4.7b}$$

where (4.7*a*) provides the orthogonal property of discrete eigenfunctions.

Lemma 1. *The following asymptotic behaviors hold:*

$$\lim_{k\to \mathrm{i}\kappa_m} \sum_{j=1,j\neq m}^{N} \frac{1}{k^2+\kappa_j^2} \int_x^{\infty} \left[\left(\phi_j^2(\xi,t)\right)_\xi \phi^2(\xi,t,k) - \phi_j^2(\xi,t)(\phi^2(\xi,t,k))_\xi\right]\mathrm{d}\xi \sim 0,$$
$$(x \to -\infty), \tag{4.8a}$$

$$\lim_{k\to \mathrm{i}\kappa_m} \frac{1}{k^2+\kappa_m^2} \int_x^{\infty} \left[\left(\phi_m^2(\xi,t)\right)_\xi \phi^2(\xi,t,k) - \phi_m^2(\xi,t)(\phi^2(\xi,t,k))_\xi\right]\mathrm{d}\xi \sim 0,$$
$$(x \to -\infty). \tag{4.8b}$$

Proof. Noting that each $\left(\phi_j^2\right)_x$ is an eigenfunction of the operator T with respect to the eigenvalue $4\kappa_j^2$ and $T\partial = \partial^3 + 4u\partial + 2u_x$ is an anti-symmetric operator with respect to the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f\cdot g\,\mathrm{d}x$ for rapidly decreasing functions f and g as $|x| \to \infty$, we have

$$\begin{aligned}
\kappa_j^2\left\langle \left(\phi_j^2\right)_x, \phi^2(x,t,\mathrm{i}\kappa_m)\right\rangle &= \tfrac{1}{4}\left\langle T\partial\phi_j^2, \phi^2(x,t,\mathrm{i}\kappa_m)\right\rangle \\
&= -\tfrac{1}{4}\left\langle \phi_j^2, T\partial\phi^2(x,t,\mathrm{i}\kappa_m)\right\rangle \\
&= -\kappa_m^2\left\langle \phi_j^2, (\phi^2(x,t,\mathrm{i}\kappa_m))_x\right\rangle \\
&= \kappa_m^2\left\langle \left(\phi_j^2\right)_x, \phi^2(x,t,\mathrm{i}\kappa_m)\right\rangle.
\end{aligned}$$

For $j \neq m$, since $\kappa_j \neq \kappa_m$, it then turns out that $\left\langle \left(\phi_j^2\right)_x, \phi^2(x,t,\mathrm{i}\kappa_m)\right\rangle = 0$, and then (4.8*a*) holds.

For (4.8*b*), we have

$$\begin{aligned}
&\lim_{k\to \mathrm{i}\kappa_m} \frac{1}{k^2+\kappa_m^2} \int_x^{\infty} \left[\left(\phi_m^2(\xi,t)\right)_\xi \phi^2(\xi,t,k) - \phi_m^2(\xi,t)(\phi^2(\xi,t,k))_\xi\right]\mathrm{d}\xi \\
&\quad = \lim_{k\to \mathrm{i}\kappa_m} \frac{1}{k} \int_x^{\infty} [\phi_m(\xi,t)(\phi(\xi,t,k))_k[\phi_{m,\xi}(\xi,t)\phi(\xi,t,k) \\
&\quad - \phi_m(\xi,t)(\phi(\xi,t,k))_\xi] + \phi_m(\xi,t)\phi(\xi,t,k)[\phi_{m,\xi}(\xi,t)(\phi(\xi,t,k))_k \\
&\quad - \phi_m(\xi,t)(\phi(\xi,t,k))_{\xi,k}]]\,\mathrm{d}\xi,
\end{aligned}$$

where we have made use of the L'Hospital's rule. Noting that (4.2) contributes

$$\lim_{k\to \mathrm{i}\kappa_m} [\phi_{m,\xi}(\xi,t)\phi(\xi,t,k) - \phi_m(\xi,t)(\phi(\xi,t,k))_\xi] = 0,$$

and also making use of (4.6*b*) we further have

$$\begin{aligned}
&\lim_{k\to \mathrm{i}\kappa_m} \frac{1}{k^2+\kappa_m^2} \int_x^{\infty} \left[\left(\phi_m^2(\xi,t)\right)_\xi \phi^2(\xi,t,k) - \phi_m^2(\xi,t)(\phi^2(\xi,t,k))_\xi\right]\mathrm{d}\xi \\
&\quad = 2\int_x^{\infty} \phi_m(\xi,t)\phi(\xi,t,\mathrm{i}\kappa_m)\left[\partial_\xi^{-1}(\phi_m(\xi,t)\phi(\xi,t,\mathrm{i}\kappa_m))\right]\mathrm{d}\xi
\end{aligned}$$

$$= 2\int_x^{\infty} \left[\left(\partial_\xi^{-1}(\phi_m(\xi,t)\phi(\xi,t,\mathrm{i}\kappa_m))\right)^2\right]_\xi \mathrm{d}\xi$$
$$= \left(\partial_\xi^{-1}(\phi_m(\xi,t)\phi(\xi,t,\mathrm{i}\kappa_m))\right)^2\Big|_{\xi=x}^{\xi=\infty},$$

where ∂_ξ^{-1} specially means $\frac{1}{2}\left(\int_{-\infty}^{\xi} - \int_{\xi}^{\infty}\right)\mathrm{d}\xi$. It is then easy to reach (4.8*b*) by taking $x \to -\infty$. Thus we complete the proof. □

Lemma 2. *The following asymptotic behaviors hold:*

$$\lim_{k\to \mathrm{i}\kappa_m} \sum_{j=1, j\neq m}^{N} \frac{1}{k^2+\kappa_j^2}\left[\frac{1}{2}\left(\phi_j^2\right)_x \phi(x,t,k) - \phi_j^2\phi_x(x,t,k)\right] \sim 0, \qquad x\to\infty, \tag{4.9a}$$

$$\lim_{k\to \mathrm{i}\kappa_m} \frac{1}{k^2+\kappa_m^2}\left[\frac{1}{2}\left(\phi_m^2\right)_x \phi(x,t,k) - \phi_m^2\phi_x(x,t,k)\right] \sim c_m(t)\,\mathrm{e}^{-\mathrm{i}\kappa_m x}\beta_m(t), \quad x\to\infty, \tag{4.9b}$$

where $\beta_m(t)$ is given by (4.2) and (4.3).

Proof. Equation (4.9*a*) is obviously valid due to both ϕ_j and $\phi_x(x,t,\mathrm{i}\kappa_m)$ tending to zero when $x\to\infty$. By using the L'Hospital's rule the lhs of (4.9*b*) reads

$$\frac{1}{2\mathrm{i}\kappa_m}\phi_m(\phi_{m,x}(\phi(x,t,\mathrm{i}\kappa_m))_k - \phi_m(\phi(x,t,\mathrm{i}\kappa_m))_{x,k}).$$

It then follows from (4.6*b*) (taking $j=m$) that

$$\text{l.h.s. (4.9}b) = \phi_m\partial^{-1}\phi_m\phi(x,t,\mathrm{i}\kappa_m) = \phi_m\sqrt{2\beta_m(t)}\partial^{-1}\phi^2(x,t,\mathrm{i}\kappa_m),$$

where we have inserted (4.2) into the integration. Finally, noting that $\phi(x,t,\mathrm{i}\kappa_m)$ is normalized, $\partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_x^{\infty}\right)\mathrm{d}x$, and also using (3.1*a*) and (4.1) we have

$$\text{l.h.s. (4.9}b) \sim c_m(t)\,\mathrm{e}^{-\mathrm{i}\kappa_m x}\beta_m(t), \qquad (x\to\infty).$$ □

It is easy to verify the following lemma.

Lemma 3. *Suppose that $\phi(x,t,k)$ is a solution of (2.1a), A and B satisfy the condition $\phi_{xxt} = \phi_{txx}$, i.e. (2.2), then*

$$P(x,t,k) = \phi_t(x,t,k) - A\phi(x,t,k) - B\phi_x(x,t,k) \tag{4.10}$$

solves (2.1a) as well.

Based on the above three lemmas, we now give time evolutions of discrete scattering data.

Theorem 1. *The discrete scattering data in (3.4) satisfy the following time evolutions: for $n=0$,*

$$\kappa_j(t) = \kappa_j(0)\,\mathrm{e}^t, \qquad c_j(t) = c_j(0)\exp\left(\frac{1}{2}t + \int_0^t \beta_j(\tau)\,\mathrm{d}\tau\right); \tag{4.11a}$$

for $n\neq 0$,

$$\begin{aligned} &\kappa_j(t) = \left(-2n\cdot 4^n t + \kappa_j^{-2n}(0)\right)^{-1/2n}, \\ &c_j(t) = c_j(0)\exp\left(\int_0^t \left[\frac{\left(n+\frac{1}{2}\right)4^n}{-2n\cdot 4^n\tau + \kappa_j^{-2n}(0)} + \beta_j(\tau)\right]\mathrm{d}\tau\right). \end{aligned} \tag{4.11b}$$

Proof. Suppose that $\overline{\phi}(x,t,k)$ is another Jost solution of (2.1*a*), which is linearly independent of $\phi(x,t,k)$. Since from lemma 3 $P(x,t,k)$ also solves (2.1*a*), there exist $\alpha(k)$ and $\beta(k)$ such that

$$\phi_t(x,t,k)+\frac{B_x}{2}\phi(x,t,k)-B\phi_x(x,t,k)=\alpha(k)\phi(x,t,k)+\beta(k)\overline{\phi}(x,t,k). \tag{4.12}$$

Noting the fact of both $\phi(x,t,k)$ and $\phi_x(x,t,k)$ tending to zero as $x\to\infty$ for $\mathrm{Im}\,k>0$, it is easily to see $\beta(k)=0$, i.e.,

$$\phi_t(x,t,k)+\frac{B_x}{2}\phi(x,t,k)-B\phi_x(x,t,k)=\alpha(k)\phi(x,t,k). \tag{4.13}$$

We now multiply (4.13) by $2\phi(x,t,k)$ and integrate them. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_x^\infty\phi^2(x,t,k)\,\mathrm{d}x+\int_x^\infty[B_x\phi^2(x,t,k)-B(\phi^2(x,t,k))_x]\,\mathrm{d}x=2\alpha(k)\int_x^\infty\phi^2(x,t,k)\,\mathrm{d}x. \tag{4.14}$$

For convenient we divide B as

$$B=B_1+B_2, \tag{4.15a}$$

and

$$B_1=(-4k^2)^nx+\sum_{j=1}^n2(-4k^2)^{n-j}\partial^{-1}T^{j-1}\qquad(xu_x+2u), \tag{4.15b}$$

$$B_2=-\sum_{j=1}^N\frac{\phi_j^2}{k^2+\kappa_j^2}. \tag{4.15c}$$

Now let us determine the value of $\alpha(\mathrm{i}\kappa_m)$. First combining (4.8*a*) and (4.8*b*) in lemma 1 just leads to

$$\lim_{k\to\mathrm{i}\kappa_m}\int_x^\infty[B_{2,x}\phi^2(x,t,k)-B_2(\phi^2(x,t,k))_x]\,\mathrm{d}x\sim0,\qquad(x\to\infty). \tag{4.16}$$

Then, noting that $\int_{-\infty}^{+\infty}\phi^2(x,t,\mathrm{i}\kappa_m)\,\mathrm{d}x=1$ we further have

$$\alpha(\mathrm{i}\kappa_m)=\frac12\int_{-\infty}^\infty[B_{1,x}\phi^2(x,t,\mathrm{i}\kappa_m)-B_1(\phi^2(x,t,\mathrm{i}\kappa_m))_x]\,\mathrm{d}x. \tag{4.17}$$

Thus, further from [22] we can reach

$$\alpha(\mathrm{i}\kappa_m)=(n+1)(2\kappa_m)^{2n},\qquad(m=1,2,\ldots,N). \tag{4.18}$$

With $\alpha(\mathrm{i}\kappa_m)$ in hand, we derive time revolutions of the discrete scattering data. We consider the limit of (4.13) with respect to $k\to\mathrm{i}\kappa_m$ and the asymptotic result under $x\to\infty$. Thanks for

$$\lim_{k\to\mathrm{i}\kappa_m}\left(\frac{B_{2,x}}{2}\phi(x,t,k)-B_2\phi_x(x,t,k)\right)\sim-c_m(t)\,\mathrm{e}^{-\kappa_mx}\beta_m(t),\qquad(x\to\infty), \tag{4.19}$$

which comes from the combination of (4.9*a*) and (4.9*b*) in lemma 2, we have

$$\begin{aligned}c_{m,t}\,\mathrm{e}^{-\kappa_mx}&+c_m(-\kappa_{m,t}x)\,\mathrm{e}^{-\kappa_mx}+\tfrac12\cdot4^n\kappa_m^{2n}c_m\,\mathrm{e}^{-\kappa_mx}-\left(4^n\kappa_m^{2n}x\right)(-c_m\kappa_m)\,\mathrm{e}^{-\kappa_mx}\\&-c_m(t)\,\mathrm{e}^{-\kappa_mx}\beta_m(t)=(n+1)4^n\kappa_m^{2n}\,\mathrm{e}^{-\kappa_mx}c_m,\qquad(m=1,2,\ldots,N),\end{aligned} \tag{4.20}$$

where we have made use of $\phi(x,t,\mathrm{i}\kappa_m) = c_m(t)\phi^+(x,t,\mathrm{i}\kappa_m)$, (4.18) and (3.1*a*) with $k = \mathrm{i}\kappa_m$. Thus we finally obtain

$$\kappa_{m,t}(t) = 4^n\kappa_m^{2n+1}(t), \qquad c_{m,t}(t) = \left[\left(n+\tfrac{1}{2}\right)4^n\kappa_m^{2n}(t) + \beta(t)\right]c_m(t), \qquad (m = 1, 2, \ldots, N). \tag{4.21}$$

The proof has been finished. □

Obviously, here the evolution of the normalization constant $c_m(t)$ is added by an extra term $\beta_m(t)c_m(t)$ compared with the one of the nonisospectral KdV hierarchy without source [22].

5. *N*-soliton solution for the nonisospectral KdVESCS hierarchy

In this section, we drive out the reflectionless potentials of the hierarchy of the nonisospectral KdVESCS (1.1).

According to the standard IST procedure given in [20], the solution of the nonisospectral KdVESCS hierarchy (1.1) can be derived in the following way:

$$u(x,t) = 2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x,t), \tag{5.1a}$$

$$\phi_j(x,t) = \sqrt{2\beta_j(t)}c_j(t)\psi^+(x,t,\mathrm{i}\kappa_j) \tag{5.1b}$$

$$= \sqrt{2\beta_j(t)}c_j(t)\left(\mathrm{e}^{-\kappa_j x} + \int_x^\infty K(x,s,t)\,\mathrm{e}^{-\kappa_j s}\,\mathrm{d}s\right), \qquad (j = 1, 2, \ldots, N), \tag{5.1c}$$

where $K(x,y,t)$ satisfies the Gel'fand–Levitan–Marchenko equation,

$$K(x,y,t) + F(x+y,t) + \int_x^\infty K(x,s,t)F(s+y,t)\,\mathrm{d}s = 0, \qquad (y > x) \tag{5.2}$$

with

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty R(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x + \sum_{j=1}^N c_j^2(t)\,\mathrm{e}^{-\kappa_j x}. \tag{5.3}$$

Particularly, when the reflection coefficient $R(t,k(t)) = 0$, (5.2) reads

$$K(x,y,t) + \sum_{m=1}^N c_m^2(t)\,\mathrm{e}^{-\kappa_m(t)(x+y)} + \sum_{m=1}^N c_m^2(t)\,\mathrm{e}^{-\kappa_m(t)y}\int_x^\infty K(x,s,t)\,\mathrm{e}^{-\kappa_m(t)s}\,\mathrm{d}s = 0. \tag{5.4}$$

Then substituting the variable separation form

$$K(x,y,t) = \sum_{l=1}^N c_l(t)h_l(x)\,\mathrm{e}^{-\kappa_l(t)y} \tag{5.5}$$

into (5.4) yields

$$K(x,x,t) = 2\frac{\mathrm{d}}{\mathrm{d}x}\ln\det D(x,t), \tag{5.6}$$

where

$$D(x,t) = (d_{ij}(x,t))_{N\times N}, \qquad d_{ij} = \delta_{ij} + \frac{1}{\kappa_i(t)+\kappa_j(t)}c_i(t)c_j(t)\,\mathrm{e}^{-(\kappa_i(t)+\kappa_j(t))x}. \tag{5.7}$$

And finally we reach

$$u(x,t) = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\det D(x,t), \tag{5.8a}$$

$$\phi_j(x,t) = \sqrt{2\beta_j(t)}\frac{\sum_{m=1}^{N} c_m(t)\,\mathrm{e}^{-\kappa_m x} D_{mj}}{\det D(x,t)}, \tag{5.8b}$$

where D_{mj} denotes the co-factors of the matrix $D(x,t)$. Thus, for different n, from theorem 1 we can recover $\kappa_j(t)$ and $c_j(t)$ from their initial value $\kappa_j(0)$ and $c_j(0)$, and further get solutions for the whole nonisospectral KdVESCS hierarchy (1.1).

We write out the explicit forms of exact solutions of the first three equations in (1.1), which correspond to $n = 0$, 1 and 2, respectively. One-soliton solutions (corresponding to $N = 1$) for these three equations are, respectively,

$$u_0(x,t) = 2\rho_1^2\,\mathrm{e}^{2t}\,\mathrm{sech}^2\,\theta_{0,1}, \qquad \phi_{0,1}(x,t) = \sqrt{\rho_1\,\mathrm{e}^t\beta_1(t)}\,\mathrm{sech}\,\theta_{0,1}, \tag{5.9a}$$

$$u_1(x,t) = \frac{2}{\rho_1 - 8t}\,\mathrm{sech}^2\,\theta_{1,1}, \qquad \phi_{1,1}(x,t) = \frac{\sqrt{\beta_1(t)}}{(\rho_1 - 8t)^{\frac{1}{4}}}\,\mathrm{sech}\,\theta_{1,1}, \tag{5.9b}$$

$$u_2(x,t) = \frac{2}{(\rho_1 - 64t)^{\frac{1}{2}}}\,\mathrm{sech}^2\,\theta_{2,1}, \qquad \phi_{2,1}(x,t) = \frac{\sqrt{\beta_1(t)}}{(\rho_1 - 64t)^{\frac{1}{8}}}\,\mathrm{sech}\,\theta_{2,1}, \tag{5.9c}$$

where we define

$$\theta_{0,j} = -xk_{0,j}(t) + \int_0^t \beta_j(\tau)\,\mathrm{d}\tau + \theta_{0,j}^{(0)}, \tag{5.10a}$$

$$\theta_{n,j} = -xk_{n,j}(t) + \int_0^t \beta_j(\tau)\,\mathrm{d}\tau - n\ln k_{n,j}(t) + \theta_{n,j}^{(0)}, \qquad (n = 1, 2), \tag{5.10b}$$

$$k_{0,j}(t) = \rho_j\,\mathrm{e}^t, \qquad k_{n,j}(t) = (\rho_j - 8^n t)^{-\frac{1}{2n}}, \qquad (n = 1, 2), \tag{5.10c}$$

and ρ_j, $\theta_{n,j}^{(0)}$ $(n = 0, 1, 2)$ are all real constants.

For $N = 2$, we have the following uniform expression for two-soliton solutions for the three equations correspond to $n = 0$, 1 and 2 in (1.1):

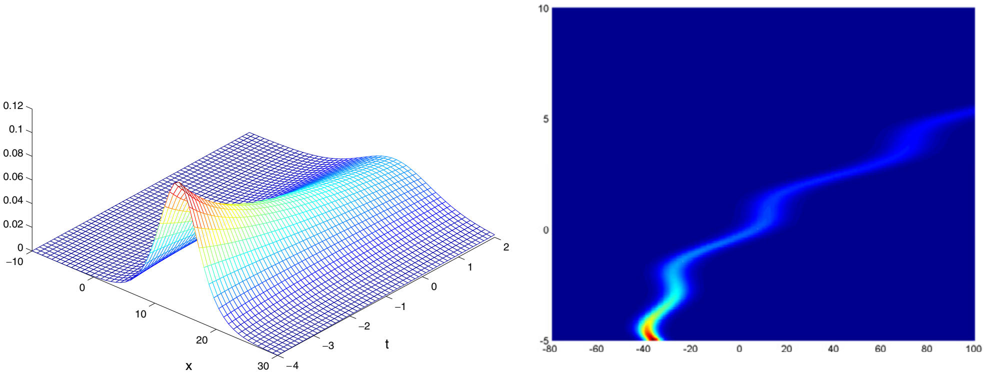
$$u_n(x,t) = 8\frac{k_{n,1}^2\,\mathrm{e}^{2\theta_{n,1}} + k_{n,2}^2\,\mathrm{e}^{2\theta_{n,2}} + 2(k_{n,1} - k_{n,2})^2\,\mathrm{e}^{2(\theta_{n,1}+\theta_{n,2})} + A_n^2\left(k_{n,2}^2\,\mathrm{e}^{4\theta_{n,1}+2\theta_{n,2}} + k_{n,1}^2\,\mathrm{e}^{2\theta_{n,1}+4\theta_{n,2}}\right)}{\left(1 + \mathrm{e}^{2\theta_{n,1}} + \mathrm{e}^{2\theta_{n,2}} + A_n^2\,\mathrm{e}^{2(\theta_{n,1}+\theta_{n,2})}\right)^2}, \tag{5.11a}$$

$$\phi_{n,1}(x,t) = \frac{2\sqrt{k_{n,1}\beta_1(t)}\,\mathrm{e}^{\theta_{n,1}}(1 + A_n\,\mathrm{e}^{2\theta_{n,2}})}{1 + \mathrm{e}^{2\theta_{n,1}} + \mathrm{e}^{2\theta_{n,2}} + A_n^2\,\mathrm{e}^{2(\theta_{n,1}+\theta_{n,2})}}, \tag{5.11b}$$

$$\phi_{n,2}(x,t) = \frac{2\sqrt{k_{n,2}\beta_2(t)}\,\mathrm{e}^{\theta_{n,2}}(1 - A_n\,\mathrm{e}^{2\theta_{n,1}})}{1 + \mathrm{e}^{2\theta_{n,1}} + \mathrm{e}^{2\theta_{n,2}} + A_n^2\,\mathrm{e}^{2(\theta_{n,1}+\theta_{n,2})}}, \tag{5.11c}$$

where $k_{n,j} \doteq k_{n,j}(t)$ and $\theta_{n,j}$ are defined as (5.10), $A_n = \frac{k_{n,1}(t)-k_{n,2}(t)}{k_{n,1}(t)+k_{n,2}(t)}$.

For convenience, we replace t with $-t$, then the hierarchy of the nonisospectral KdVESCS reads



(*a*) (*b*)

Figure 1. The shape and motion of one soliton of the nonisospectral KdVESCS. (*a*) A stationary soliton given by (5.9*b*) for $\rho_1 = 50$, $\beta_1(t) = -\frac{4}{\rho_1+8t} - \sqrt{\rho_1}(\rho_1+8t)^{-\frac{3}{2}}(2\ln\rho_1 + 4\theta_{1,1}^{(0)})$ and $\theta_{1,1}^{(0)} = 0$. (*b*) The density plot of a moving soliton given by (5.9*b*) for $\rho_1 = 50$, $\beta_1(t) = 2 - 2\sin 2t$ and $\theta_{1,1}^{(0)} = 1$, $x \in [-80, 100]$, $t \in [-5, 10]$. The blue area denotes zero value and bright strap denotes positive soliton.

$$u_t = -T^n(xu_x + 2u) - 2\sum_{j=1}^{N}\left(\phi_j^2\right)_x, \qquad (n = 0, 1, 2, \ldots), \tag{5.12a}$$

$$\phi_{j,xx} = (\lambda_j - u)\phi_j, \qquad (j = 1, 2, \ldots, N). \tag{5.12b}$$

In the following, we consider the dynamics of the solution for the nonisospectral KdVESCS. Equation (5.9*b*) shows that a soliton travels with a decay amplitude $\frac{2}{\rho_1+8t}$, and the time-dependent top trace

$$x(t) = \sqrt{\rho_1 + 8t}\left(\int_0^t \beta_1(\tau)\,\mathrm{d}\tau + \frac{1}{2}\ln(\rho_1 + 8t) + \theta_{0,1}^{(0)}\right), \tag{5.13}$$

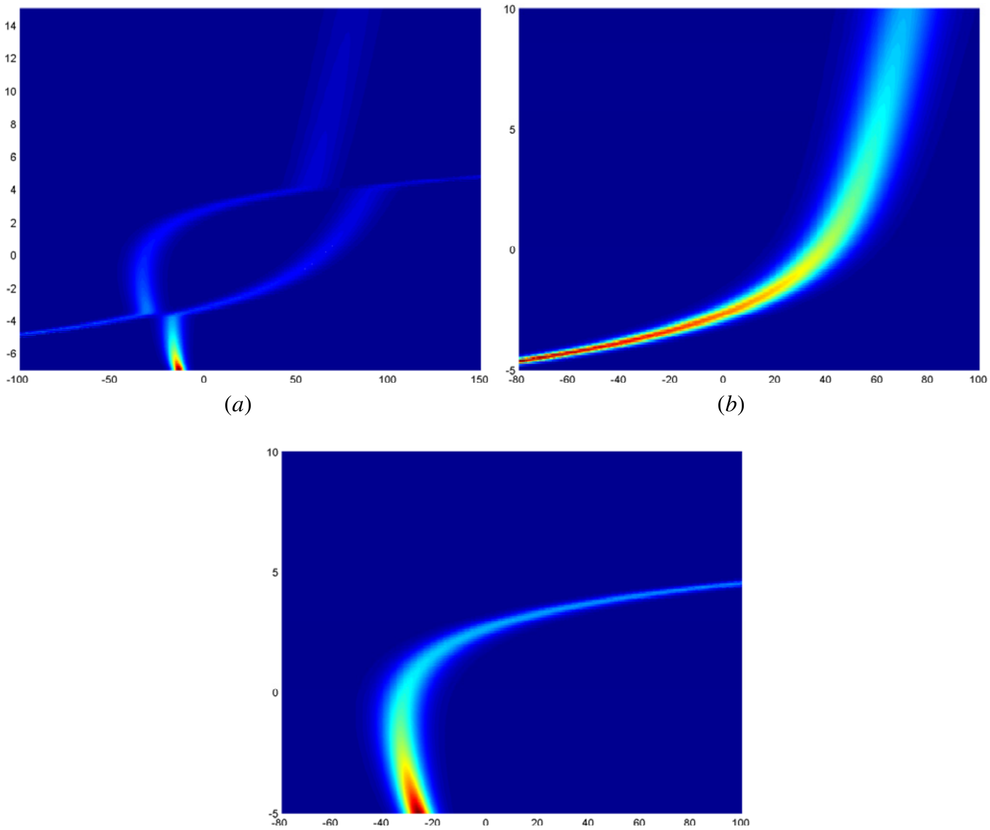
or the speed of the propagate

$$x'(t) = \frac{(\rho_1 + 8t)\beta_1(t) + 4 + 4\left(\int_0^t \beta_1(\tau)\,\mathrm{d}\tau + \frac{1}{2}\ln(\rho_1 + 8t) + \theta_{0,1}^{(0)}\right)}{\sqrt{\rho_1 + 8t}}. \tag{5.14}$$

The non-negative continuous function $\beta_1(t)$ acts as source which affects the speed of the soliton but not the shape. When we take $\rho_1 > 0$, $\theta_{0,1}^{(0)} \geqslant 0$ and $\beta_1(t) = -\frac{4}{\rho_1+8t} - \sqrt{\rho_1}(\rho_1 + 8t)^{-\frac{3}{2}}\left(2\ln\rho_1 + 4\theta_{0,1}^{(0)}\right)$, we can obtain the stationary soliton. In this case, $x(t) \equiv \sqrt{\rho_1}\left(\frac{1}{2}\ln\rho_1 + \theta_{0,1}^{(0)}\right)$ and $x'(t) \equiv 0$. Figure 1 describes the shape and motion of one soliton.

Taking $n = 1$ in (5.11), we derive the two-soliton solution for the nonisospectral KdVESCS. First, we suppose that $\rho_2 > \rho_1 > 0$, and $t > -\frac{\rho_1}{8}$. In this case, it is not easy to analytically investigate two-soliton interactions, but they do have elastic scattering. Figure 2(*a*) shows the elastic interactions of two solitons with decay amplitudes. For comparison, we give the two corresponding single solitons in figures 2(*b*) and (*c*). Especially, when we take $\rho_1 = \rho_2 > 0$, we get the degenerate two-soliton solutions. In this case, we have

$$u_1(x,t) = \frac{8(\mathrm{e}^{2\theta_{1,1}} + \mathrm{e}^{2\theta_{1,2}})}{(\rho_1 + 8t)(1 + \mathrm{e}^{2\theta_{1,1}} + \mathrm{e}^{2\theta_{1,2}})^2}, \tag{5.15}$$



(*a*) (*b*) (*c*)

Figure 2. One-soliton behaviors and two-soliton interactions of the nonisospectral KdVESCS. (*a*) The density plot of the two-soliton solution given by (5.11) ($n = 1$) for $\rho_1 = 65, \rho_2 = 85, \beta_1(t) = 0.6\,\mathrm{e}^{0.6t}, \beta_2(t) = 0.6\,\mathrm{e}^{-0.6t}, \theta_{1,1}^{(0)} = 6, \theta_{1,2}^{(0)} = -2$, and $x \in [-100, 150], t \in [-7, 15]$. (*b*) The density of corresponding one soliton by (5.9*b*) for $\rho_1 = 65, \beta_1(t) = 0.6\,\mathrm{e}^{0.6t}, \theta_{1,1}^{(0)} = 6$, and$x \in [-100, 150], t \in [-7, 15]$. (*c*) The density of corresponding one soliton by (5.9*b*) for $\rho_2 = 85, \beta_2(t) = 0.6\,\mathrm{e}^{-0.6t}, \theta_{1,2}^{(0)} = -2$, and$x \in [-100, 150], t \in [-7, 15]$. The blue area denotes zero value and bright strap denotes positive soliton.

where

$$\theta_{1,j} = \frac{x}{\sqrt{\rho_1 + 8t}} - \int_0^t \beta_j(\tau)\,\mathrm{d}\tau - \frac{1}{2}\ln(\rho_1 + 8t) + \theta_{1,j}^{(0)}, \qquad j = 1, 2.$$

Figures 3(*a*) and (*b*) describe that a single soliton is ‘disturbed’ by invisible ‘ghost’ soliton. In figure 3(*c*), a soliton travels first with its original source and then suddenly with another different source.

Then we investigate the asymptotic behaviors for more detail about such degenerate case. Suppose that $\beta_j(t)$ satisfy

$$\int_0^t (\beta_1(t) - \beta_2(t))\,\mathrm{d}t \to -\infty, \qquad \text{as} \quad t \to +\infty \tag{5.16}$$

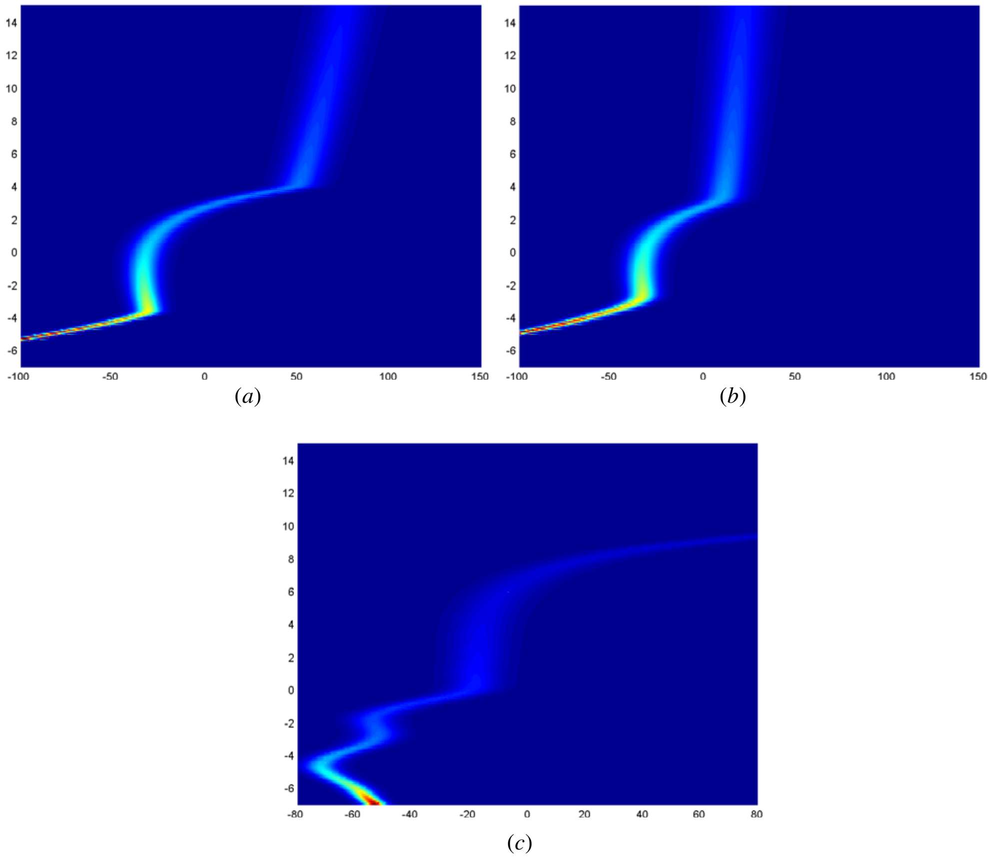


Figure 3. The degenerate two-soliton solution ($\rho_1 = \rho_2$) of the nonisospectral KdVESCS. (*a*) The density plot of the degenerate two-soliton solution given by (5.15) for $\rho_1 = \rho_2 = 65$, $\beta_1(t) = 0.6\,\mathrm{e}^{0.6t}$, $\beta_2(t) = 0.6\,\mathrm{e}^{-0.6t}$, $\theta_{1,1}^{(0)} = 6$, $\theta_{1,2}^{(0)} = -2$, and $x \in [-100, 150]$, $t \in [-7, 15]$. (*b*) The density of the degenerate two-soliton by (5.15) with same parameters as (*a*) except $\theta_{1,2}^{(0)} = 2$ instead of -2. Thus one can see the existence of $\theta_{1,2}$-soliton by comparing (*a*) and (*b*). (*c*) The density plot of the degenerate two-soliton solution given by (5.15) for $\rho_1 = \rho_2 = 65$, $\beta_1(t) = 0.02\,\mathrm{e}^{0.6t}$, $\beta_2(t) = 2 - 2sin2t$, $\theta_{1,1}^{(0)} = 4.2$, $\theta_{1,2}^{(0)} = 4$, and $x \in [-80, 80]$, $t \in [-7, 15]$. The blue area denotes zero value and bright strap denotes positive soliton.

and the two solitons involved in the degenerate case are called $\theta_{1,1}$-soliton and $\theta_{1,2}$-soliton, respectively. We consider the coordinate frame co-moving with $\theta_{1,1}$-soliton,

$$\left(X = \theta_{1,1} = \frac{x}{\sqrt{\rho_1 + 8t}} - \int_0^t \beta_1(\tau)\,\mathrm{d}\tau - \frac{1}{2}\ln(\rho_1 + 8t) + \theta_{1,1}^{(0)}, t\right), \tag{5.17}$$

where $\theta_{1,1}$ stay zero but

$$\theta_{1,2} = \theta_{1,1} + \int_0^t \beta_1(\tau)\,\mathrm{d}\tau + \theta_{1,2}^{(0)} - \theta_{1,1}^{(0)} \to -\infty, \qquad \text{as} \quad t \to +\infty \tag{5.18}$$

due to (5.22). This further gives

$$u_1(x,t) \to \frac{2}{\rho_1 + 8t}\,\mathrm{sech}^2\,\theta_{1,1}, \qquad t \to +\infty. \tag{5.19}$$

Similarly, under the frame

$$\left(Y = \theta_{1,2} = \frac{x}{\sqrt{\rho_2 + 8t}} - \int_0^t \beta_2(\tau)\,\mathrm{d}\tau - \frac{1}{2}\ln(\rho_2 + 8t) + \theta_{1,2}^{(0)}, t\right), \tag{5.20}$$

which co-moves with $\theta_{1,2}$-soliton (letting $\theta_{1,2}$ stay zero), we have

$$u \to 0, \qquad t \to +\infty. \tag{5.21}$$

Thus we conclude that for the final states of two solitons involved in the degenerate case, one exists but another disappears under condition (5.22). Obviously, similar result holds when

$$\int_0^t (\beta_1(t) - \beta_2(t))\,\mathrm{d}t \to +\infty, \qquad \text{as} \quad t \to +\infty. \tag{5.22}$$

6. Conclusion

We have derived the hierarchy of the nonisospectral KdVESCS, and obtain N-soliton solutions of the hierarchy via the IST. The key step is to determine time revolutions of scattering data. Since the source part $\sum_{j=1}^{N} \frac{\phi_j^2}{k^2+\kappa_j^2}$ in the Lax pair can generate singularities when k tends to $\mathrm{i}\kappa_j$, we have developed two lemmas (lemmas 1 and 2) to deal with the singularities and further get the time revolutions of scattering data. We have described in detail our treatment of the singularities in the paper and the treatment is slightly different from [8]. Our approach can apply to other $(1 + 1)$-dimensional soliton hierarchy with self-consistent sources.

Acknowledgments

The authors are very grateful to the referees for their invaluable comments. This project is supported by the National Natural Science Foundation of China (nos 10371070 and 10671121), the Foundation of Shanghai Education Committee for Shanghai Prospective Excellent Young Teachers, Shanghai Leading Academic Discipline Project (no J50101) and the President Foundation of East China Institute of Technology (no DHXK0810).

References

[1] Mel'nikov V K 1989 Capture and confinement of solitons in nolinear integrable systems *Commun. Math. Phys.* **120** 451–68

[2] Mel'nikov V K 1989 Interaction of solitary waves in the system described by the KP equation with a self-consistent sources *Commun. Math. Phys.* **126** 201–15

[3] Mel'nikov V K 1992 Integration method of the nonlinear Schrödinger equation with a self-consistent source *Inverse Problems* **8** 133–47

[4] Leon J and Latifi A 1990 Solutions of an initial-boundary value problem for coupled non-linear waves *J. Phys. A: Math. Gen.* **23** 1385–403

[5] Zhang D J, Bi J B and Hao H H 2006 Modified KdV equation with self-consistent sources in non-uniform media and soliton dynamics *J. Phys. A: Math. Gen.* **39** 14627–48

[6] Bi J B and Zhang D J 2007 Sine-Gordon equation with self-consistent sources in non-uniform media *Commun. Theor. Phys.* **48** 971–8

[7] Zeng Y, Ma W and Lin R 2000 Integration of the soliton hierarchy with self-consistent sources *J. Math. Phys.* **41** 5453–89

[8] Lin R, Zeng Y and Ma W 2001 Solving the KdV hierarchy with self-consistent sources by inverse scattering method *Physica* A **291** 287–98

[9] Zeng Y, Ma W and Shao Y 2001 Two binary Darboux transformations for the KdV hierarchy with self-consistent sources *J. Math. Phys.* **42** 2113–28

[10] Zhang D J 2002 The *N*-soliton solutions for the modified KdV equation with self-consistent sources *J. Phys. Soc. Japan* **71** 2649–56
[11] Zhang D J and Chen D Y 2003 The *N*-soliton solutions of the sine-Gordon equation with self-consistent sources *Physica* A **321** 467–81
[12] Zhang D J 2003 The *N*-soliton solutions of some soliton equations with self-consistent sources *Chaos Solitons Fractals* **18** 31–43
[13] Deng S F, Chen D Y and Zhang D J 2003 The multisoliton solutions of the KP equation with self-consistent sources *J. Phys. Soc. Japan* **72** 2184–92
[14] Hu X B and Wang H Y 2006 Construction of dKP and BKP equations with self-consistent sources *Inverse Problems* **22** 1903–20
[15] Chen H H and Liu C S 1976 Solitons in nonuniform media *Phys. Rev. Lett.* **37** 693–7
[16] Hirota R and Satsuma J 1976 *N*-soliton solution of the KdV equation with loss and nonuniformity terms *J. Phys. Soc. Japan* **41** 2141–2
[17] Gupta M R 1979 Exact inverse scattering solution of a nonlinear evolution equation in a nonuniform media *Phys. Lett.* A **72** 420–2
[18] Fuchssteiner B 1983 Master symmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations *Prog. Theor. Phys.* **70** 1508–22
[19] W X Ma and Fuchssteiner B 1999 Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations *J. Math. Phys.* **40** 2400–18
[20] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
[21] Chen D Y 2006 *Introduction of Soliton Theory* (Beijing: Science Press)
[22] Ning T K 2004 Soliton-like solutions for a nonisospectral KdV hierarchy *Chaos Solitons Fractals* **21** 395–401